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INTERPRETATIONS OF THE IDENTICAL RELATIONS BETWEEN THE DETERMINANTS OF AN ARRAY.*

By R. P. BAKER. University of Iowa.

§1. $n=2, m=4$

(1) If the elements represent homogeneous coördinates of four points on a line, the 2×2 minors are proportional to the distances between pairs of points, and the identical relation in them gives Ptolemy's theorem.

$$(12)(34) - (13)(24) + (14)(23) = 0.$$

The relations lineo-linear in elements and determinants such as,

$$\begin{aligned} a_1(a_2b_3) + a_2(a_3b_1) + a_3(a_1b_2) &= 0, \\ b_1(a_2b_3) + b_2(a_3b_1) + b_3(a_1b_2) &= 0, \end{aligned}$$

give by linear combination Ptolemy's theorem for the points 1, 2, 3 and any other point in the line, or if this point be at infinity the identical relation between the distances of the three points 1, 2, 3.

(2) The elements represent homogeneous coördinates of two points A, B , in space. The determinants are the line coördinates of AB , and the identical relation the "fundamental relation" between them. The relations

$$z_1(a_2b_3) + z_2(a_3b_1) + z_3(a_1b_2) = 0 \quad (z=a, b)$$

may be taken as a relation between the distances of any point in the line and the moments of the line with respect to the three planes which pass through a vertex.

If z_1, z_2, z_3 are current coördinates the relation is the equation of a plane through a vertex and the line.

§2. $n=2, m=5$.

The general formula gives three independent relations. For any eight elements and their six determinants we have the identity given in the case $n=2, m=4$. Thus we have the five relations,

$$\begin{aligned} A_1; (23)(45) - (24)(35) + (25)(34) &= 0, \\ A_2; (34)(51) - (35)(41) + (31)(45) &= 0, \\ A_3; (45)(12) - (41)(52) + (42)(51) &= 0, \\ A_4; (51)(23) - (52)(13) + (53)(12) &= 0, \\ A_5; (12)(34) - (13)(24) + (14)(23) &= 0. \end{aligned}$$

*Sequel to a paper with similar title in the January number of the MONTHLY.

These relations are dependent in any case on some three of them, but the same three cannot always be taken as independent. There exist, in fact, five identities in the determinants:

$$\begin{aligned} A'_1; (12)A_2 + (13)A_3 - (41)A_4 - (51)A_5 &= 0, \\ A'_2; (23)A_3 + (24)A_4 - (52)A_5 - (12)A_1 &= 0, \\ A'_3; (34)A_4 + (35)A_5 - (13)A_1 - (23)A_2 &= 0, \\ A'_4; (45)A_5 + (41)A_1 - (24)A_2 - (34)A_3 &= 0, \\ A'_5; (51)A_1 + (52)A_2 - (35)A_3 - (45)A_4 &= 0. \end{aligned}$$

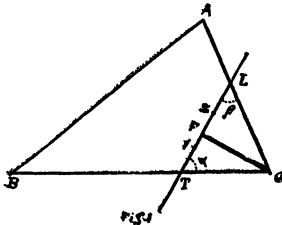
The trivial case of all vanishing determinants being excluded, suppose that (12) is not zero. Then if A_3, A_4, A_5 , vanish the relations A'_1, A'_2 , show that A_1, A_2 , also vanish, but the vanishing of any other three of the A 's will not necessarily entail the vanishing of the remaining pair. There are also twenty lineo-linear relations.

The geometrical interpretation is similar to that in the case of a 2×4 matrix.

TRILINEAR COORDINATES.

If the fundamental triangle be ABC and a_1 is the perpendicular distance from the point 1 to the side BC , to interpret $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ drop a perpendicular from C to the line 12 and let 12 meet BC at the angle α , AC at the angle β .

$$\begin{aligned} \text{Then } 1T &= a_1 / \sin \alpha, & 2T &= a_2 / \sin \alpha, \\ 1L &= b_1 / \sin \beta, & 2L &= b_2 / \sin \beta, \end{aligned}$$



$$\begin{aligned} \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} &= \sin \alpha \sin \beta \begin{vmatrix} 1T & 1L \\ 2T & 2L \end{vmatrix} = \sin \alpha \sin \beta \begin{vmatrix} 1T & TL \\ 2T & TL \end{vmatrix} \\ &= TL \sin \alpha \sin \beta \overline{(12)}. \end{aligned}$$

$$\text{Now } TL = TF - FL = CF(\cot \alpha + \cot \beta).$$

$$\therefore \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = CF \sin C \overline{(12)} = (\text{moment of line 12}$$

with respect to C). $\sin C$.

To interpret $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, we have the identities,

$$\begin{aligned} BC.a_1 + CA.b_1 + AB.c_1 &= 2\Delta, \\ BC.a_2 + CA.b_2 + AB.c_2 &= 2\Delta, \\ BC.a_3 + CA.b_3 + AB.c_3 &= 2\Delta, \end{aligned}$$

$$\text{whence } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \frac{2\Delta}{BC} \begin{vmatrix} b_1 & c_1 & 1 \\ b_2 & c_2 & 1 \\ b_3 & c_3 & 1 \end{vmatrix} = \frac{2\Delta}{BC} [(b_1 c_2) + (b_2 c_3) + (b_3 c_1)]$$

$$= \frac{2 \cdot \Delta}{BC} [\bar{12} \cdot \sin A + \bar{23} \cdot \sin A + \bar{31} \cdot \sin A],$$

where $\bar{12}$ denotes the moment of the line with respect to A . That is

$$(a_1 \ b_2 \ c_3) = \frac{2 \cdot \Delta \cdot \sin A}{BC} \text{ Area } 123 = \frac{4 \cdot \Delta^2}{AB \cdot BC \cdot CA} \text{ Area } 123.$$

§3. $n=3, m=4$.

There are no relations between the determinants; there are three relations lineo-linear in the elements and determinants and twelve relations lineo-linear in the determinants and their 2×2 minors. To interpret in R_2 ,

(1) The lineo-linear relations between elements and determinants

$$x_1(a_2 b_3 c_4) - x_2(a_3 b_4 c_1) + x_3(a_4 b_1 c_2) - x_4(a_1 b_2 c_3) = 0 \quad [x = (a, b, c)].$$

The determinants are proportional to the areas of the triangles, the elements to the distances of the points from the sides of the reference triangle.

The relation being homogeneous in rows and columns proportionality factors need not be considered.

Making the usual convention as to signs of triangular areas, we have the theorem:

“In a complete quadrangle the sum of the products of the areas of the triangles into the distances of the remaining vertices from any line is zero.”

If the line is at infinity this reduces to the areal identity $234 \cdot -341 + 412 \cdot -123 = 0$.

(2) The relation,

$$(a_1 b_2)(a_3 b_4 c_1) - (a_1 b_3)(a_2 b_4 c_1) + (a_1 b_4)(a_2 b_3 c_1) = 0.$$

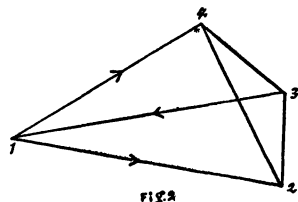
Proportionality factors being neglected as before, $(a_1 b_2)$ is the moment of the line 12 with respect to C multiplied by $\sin C$. Also $(a_1 b_2 c_3)$ is proportional to the area of the triangle 123.

The relation gives,

$$\text{Moment } 12 \times \text{Area}(341) - \text{Moment } 13 \times \text{Area}(241) + \text{Moment } 14 \times \text{Area}(231) = 0,$$

which may be interpreted as follows:

“If three concurrent forces 12, 13, 14, be proportional to the length of the lines 12, 13, 14, and to the areas of the triangles 341, 421, 231 jointly and have their signs determined by the directions in which the triangles are described they are in equilibrium.”



In the case of 1234 being a parallelogram this reduces to the parallelogram of forces.

Or, expressing the areas of the triangles by the product of adjacent sides and sine of included angle, and the moments by the product of lengths of line and perpendicular, and expressing the perpendicular by the product of the line $C1$ and the sine of the proper angle we have, if angle $C12=P$, $213=Q$, $314=R$, and the line factors, now all common, be divided out,

$$\sin(P+Q+R)\sin Q - \sin(P+Q)\sin(Q+R) + \sin P \sin R = 0,$$

a trigonometrical identity which reduces to the various forms of the addition theorem when $Q=\frac{1}{2}\pi$, $P=\frac{1}{2}\pi$.

NOTE ON FINDING THE COMPLEMENTARY FUNCTION OF A LINEAR DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS WHEN THE AUXILIARY EQUATION HAS EQUAL ROOTS.

By B. F. FINKEL.

The following method, which is the one virtually implied though not expressly stated in Forsyth's *Differential Equations*, Art. 44, p. 64, has the advantage of simplicity and elegance of presentation, and is worthy, therefore, of being better known. Dr. G. E. Fisher, of the University of Pennsylvania, has used the method for some years in his classes. Dr. Fisher does not claim that the method is original with him, neither does he remember from whence the suggestion came to him.

As I am not aware that it has been published elsewhere, at least in none of the text books on the subject that I have examined, it is thought that its appearance in the MONTHLY will be helpful to many of its readers.

Let $y = Ae^{ax} + Be^{\beta x} + \dots + Le^{lx}$ be the *Complementary Function* of the general linear differential equation

$$\frac{d^n y}{dx^n} + A_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + A_{n-1} \frac{dy}{dx} + A_n y = V.$$

Now if two of the roots are equal, as, for example, $a=\beta$, we have

$$y = (A+B)e^{ax} + Cey^x + \dots + Le^{lx} = A_1 e^{ax} + Cey^x + \dots + Le^{lx}.*$$

We thus lack an arbitrary constant, there now being only $n-1$. In order to obtain the complete primitive, let us suppose, for the moment, that a and β are different, and that $\beta-a=h$. A particular solution would then be

*Forsyth's *Differential Equations*, 3rd edition, p. 64.